Chapter Five

Fast Fourier Transform

1-1 Introduction

We have studied how to obtain DFT of a sequence by using direct computation. Basically, the direct computation of DFT requires large number of computations. So more processing time is required.

For the computation of N-point DFT, N^2 complex multiplications and $N^2 - N$ complex additions are required. If the value of N is large then the number of computations will go into lakhs. This proves inefficiency of direct DFT computation.

In 1965, Cooley and Tukey developed very efficient algorithm to implement the DFT. This algorithm is called as Fast Fourier Transform (FFT). These FFT algorithms are very efficient in terms of computations. By using these algorithms, number of arithmetic operations involved in the computation of DFT are greatly reduced.

Different FFT algorithms are available : out of which Radix-2 FFT algorithm is most important FFT algorithm.

(1) Radix-2 Decimation in Time (DIT) algorithm.

(2) Radix-2 Decimation in Frequency (DIF) algorithm.

$$\boxed{\therefore\ W_N^{k+N} = W_N^k} \qquad \boxed{\therefore\ W_N^{k+\frac{N}{2}} = -W_N^k}$$

$$\boxed{\therefore\ W_{N/2} = W_N^2}$$

Radix-2 Decimation In Time (DIT) Algorithm (DIT FFT) :

To decimate means to break into parts. Thus DIT indicates dividing (splitting) the sequence in time domain. The different stages of decimation are as follows :

First stage of decimation :

Let $x(n)$ be the given input sequence containing 'N' samples. Now for decimation in time we will divide $x(n)$ into even and odd sequences.

$$\therefore \quad x(n) = f_1(m) + f_2(m) \qquad \text{...(1)}$$

Here $f_1(m)$ is even sequence and $f_2(m)$ is odd sequence

$$\therefore \quad f_1(m) = x(2n), \qquad m = 0, 1, \ldots \frac{N}{2} - 1 \qquad \text{...(2)}$$

$$\text{and} \quad f_2(m) = x(2n+1), \qquad m = 0, 1, \ldots, \frac{N}{2} - 1 \qquad \text{..(3)}$$

Input sequence $x(n)$ has 'N' samples. So after decimation; $f_1(m)$ and $f_2(m)$ will contain $\frac{N}{2}$ samples.

Now according to the definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad \text{...(4)}$$

Since we have divided $x(n)$ into two parts; we can write separate summation for even and odd sequences as follows :

$$X(k) = \sum_{n \text{ even}} x(n) W_N^{k} + \sum_{n \text{ odd}} x(n) W_N^{kn} \qquad \text{...(5)}$$

The first summation represents even sequence. So we will put $n = 2m$ in first summation. While the second summation represents odd sequence, so we will put $n = (2m+1)$ in second summation. Since even and odd sequences contain $\frac{N}{2}$ samples each; the limits of summation will be from $m = 0$ to $\frac{N}{2} - 1$.

$$\therefore \quad X(k) = \sum_{m=0}^{\frac{N}{2}-1} x(2m)\, W_N^{2km} + \sum_{m=0}^{\frac{N}{2}-1} x(2m+1)\, W_N^{k(2m+1)} \qquad ...(6)$$

But $x(2m)$ is even sequence, so it is $f_1(m)$ and $x(2m+1)$ is odd sequence, so it is $f_2(m)$.

$$\therefore \quad X(k) = \sum_{m=0}^{\frac{N}{2}-1} f_1(m)\, W_N^{2km} + \sum_{m=0}^{\frac{N}{2}-1} f_2(m)\, W_N^{2km} \cdot W_N^{k}$$

$$\therefore \quad X(k) = \sum_{m=0}^{\frac{N}{2}-1} f_1(m) \left(W_N^2\right)^{km} + W_N^k \sum_{m=0}^{\frac{N}{2}-1} f_2(m) \left(W_N^2\right)^{km} \qquad ...(7)$$

Now we have $W_N^2 = W_{N/2}$

$$\therefore \quad X(k) = \sum_{m=0}^{\frac{N}{2}-1} f_1(m)\, W_{N/2}^{km} + W_N^k \sum_{m=0}^{\frac{N}{2}-1} f_2(m)\, W_{N/2}^{km} \qquad ...(8)$$

Comparing each summation with the definition of DFT,

$$X(k) = F_1(k) + W_N^k\, F_2(k), \quad K = 0, 1, ..., N-1 \qquad ...(9)$$

We will consider an example of 8 point DFT. That means $N = 8$.

Here $F_1(k)$ is $\frac{N}{2}$ point DFT of $f_1(m)$ and $F_2(k)$ is $\frac{N}{2}$ point DFT of $f_2(m)$. That means $F_1(k)$ and $F_2(k)$ are 4-point DFTs.

Equation (9) indicates that $F_2(k)$ is multiplied by W_N^k and it is added with $F_1(k)$, to obtain (4 + 4) i.e. 8-point DFT. Graphically Equation (9) represented as shown in Fig. G-1.

Remember that in Equation (9), K varies from 0 to N – 1 (i.e. 0 to 7 for 8 point DFT).

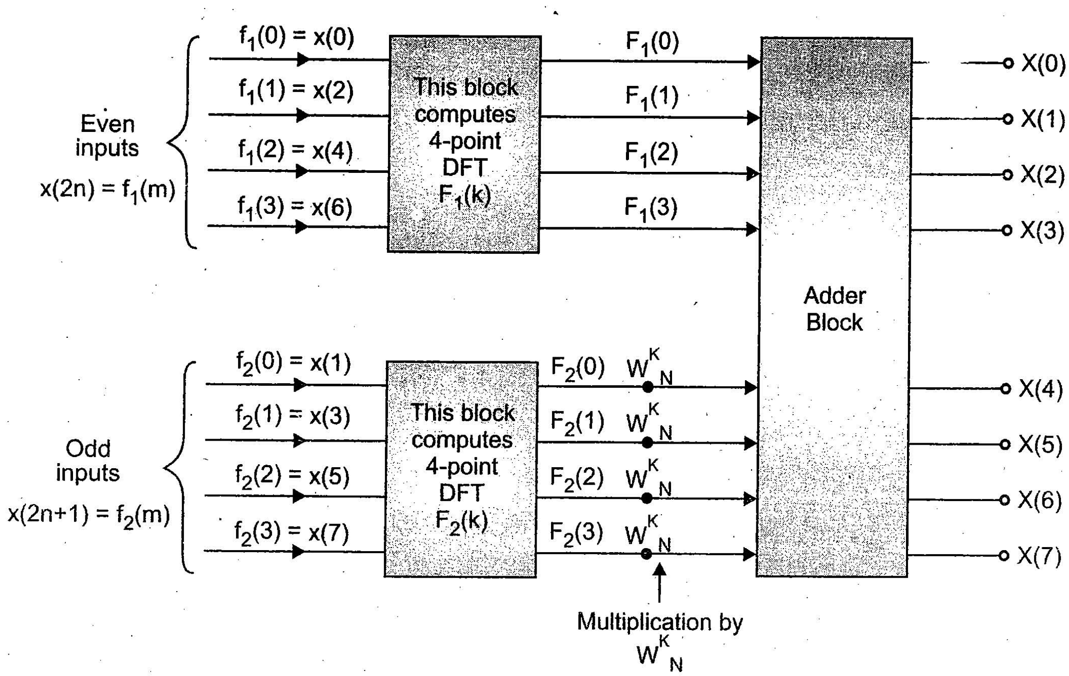


Fig. G-1 : Graphical representation of $X(k) = F_1(k) + W_N^k F_2(k)$

Now $F_1(k)$ and $F_2(k)$ are 4-point $\left(\frac{N}{2}\right)$ DFTs. They are periodic with period $\frac{N}{2}$. Using periodicity property of DFT we can write,

$$F_1\left(k+\frac{N}{2}\right) = F_1(k) \qquad \text{...(10)}$$

and $$F_2\left(k+\frac{N}{2}\right) = F_2(k) \qquad \text{...(11)}$$

Replacing k by $k+\frac{N}{2}$ in Equation (9) we get,

$$X\left(k+\frac{N}{2}\right) = F_1\left(k+\frac{N}{2}\right) + W_N^{k+\frac{N}{2}} F_2\left(k+\frac{N}{2}\right) \qquad \text{...(12)}$$

Now we have, $$W_N^{k+\frac{N}{2}} = -W_N^k$$

$$\therefore \quad X\left(k+\frac{N}{2}\right) = F_1\left(k+\frac{N}{2}\right) - W_N^k \, F_2\left(k+\frac{N}{2}\right) \qquad \text{...(13)}$$

Using Equations (10) and (11) we get,

$$X\left(k+\frac{N}{2}\right) = F_1(k) - W_N^k F_2(k) \qquad \text{....(14)}$$

Here $X(k)$ is 'N' point DFT. We can take $k = 0$ to $\frac{N}{2} - 1$ then, by using Equations (9) and (14) we can obtain combined N-point DFT.

$$\therefore \quad X(k) = F_1(k) + W_N^k F_2(k), \quad k = 0, 1 \ldots \frac{N}{2} - 1 \qquad \ldots(15)$$

$$\text{and} \quad X\left(k + \frac{N}{2}\right) = F_1(k) - W_N^k F_2(k), \quad k = 0, 1, \ldots \frac{N}{2} - 1 \qquad \ldots(16)$$

We are considering an example of 8 point DFT (N = 8). So in Equations (15) and (16), k varies from 0 to 3. Now putting k = 0 to 3 in Equations (15) and (16) we get,

$$\left.\begin{aligned} X(0) &= F_1(0) + W_N^0 F_2(0) \\ X(1) &= F_1(1) + W_N^1 F_2(1) \\ X(2) &= F_1(2) + W_N^2 F_2(2) \\ X(3) &= F_1(3) + W_N^3 F_2(3) \end{aligned}\right\} \qquad \ldots(17)$$

and

$$\left.\begin{aligned} X(0+4) = X(4) &= F_1(0) - W_N^0 F_2(0) \\ X(1+4) = X(5) &= F_1(1) - W_N^1 F_2(1) \\ X(2+4) = X(6) &= F_1(2) - W_N^2 F_2(2) \\ X(3+4) = X(7) &= F_1(3) - W_N^3 F_2(3) \end{aligned}\right\} \qquad \ldots(18)$$

The graphical representation of first stage of decimation for 8 point DFT is as shown in Fig. G-2.

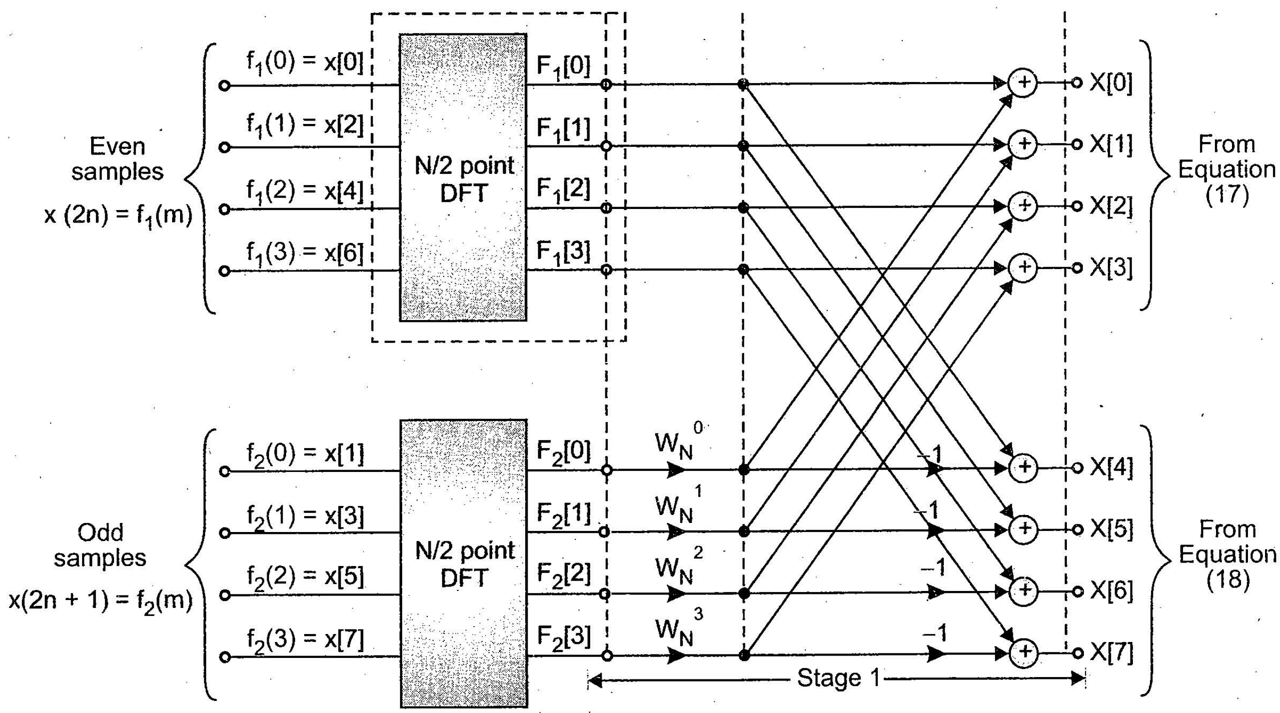


Fig. G-2 : First stage of decimation

In Fig. G-2, input sequences are

$$f_1(m) = x(2n) = \{x(0), x(2), x(4), x(6)\} \quad \text{...(19)}$$

and $$f_2(m) = x(2n+1) = \{x(1), x(3), x(5), x(7)\} \quad \text{...(20)}$$

That means each sequence contains $\frac{N}{2}$ samples.

Second stage of decimation :

In the first stage of decimation; we obtained the sequences of length $\frac{N}{2}$. That means for 8-point DFT (N = 8); the length of each sequence is '4' as given by Equations (19) and (20). We discussed that we have to continue this process till we get '2' point sequence.

We can further decimate $f_1(m)$ into even and odd samples. Let $g_{11}(n) = f_1(2m)$, which contains even samples and let $g_{12}(n) = f_1(2m+1)$, which contains odd samples of $f_1(m)$.

Note that here range of 'n' and 'm' is from 0 to $\frac{N}{4} - 1$.

Now recall Equations (15) and (16). We obtained sequences $X(k)$ and $X\left(k+\frac{N}{2}\right)$ from $F_1(k)$ and $F_2(k)$. The length of each sequence was $\frac{N}{2}$. Here in the second stage of decimation; we are further dividing the sequences into even and odd parts. So similar to Equations (15) and (16) we can write; For $F_1(k)$,

$$F_1(k) = G_{11}(k) + W_{N/2}^{k} G_{12}(k) \qquad k = 0, 1, \ldots \frac{N}{4} - 1 \quad \text{...(21)}$$

and $$F_1\left(k+\frac{N}{4}\right) = G_{11}(k) - W_{N/2}^{k} G_{12}(k) \qquad k = 0, 1, \ldots \frac{N}{4} - 1 \quad \text{...(22)}$$

Thus, for N = 8 we have the range of K, from K = 0 to K = 1. Here $G_{11}(k)$ is DFT of $g_{11}(n)$ and $G_{12}(k)$ is DFT of $g_{12}(n)$. Now putting the values of 'K' in Equation (21) we get,

$$\left.\begin{aligned} F_1(0) &= G_{11}(0) + W_{N/2}^{0} G_{12}(0) \\ F_1(1) &= G_{11}(1) + W_{N/2}^{1} G_{12}(1) \end{aligned}\right\} \quad \text{...(23)}$$

Similarly from Equation (22) we get,

$$\left.\begin{aligned} F_1\left(0+\frac{8}{4}\right) &= F_1(2) = G_{11}(0) - W_{N/2}^{0} G_{12}(0) \\ F_1\left(1+\frac{8}{4}\right) &= F_1(3) = G_{11}(1) - W_{N/2}^{1} G_{12}(1) \end{aligned}\right\} \quad \text{...(24)}$$

Here the values of K are '0' and 1. That means it is 2-point DFT. Thus Equations (23) and (24) shows that we can obtain 4-point DFT by combining two 2-point DFTs. The graphical representation is shown in Fig. G-3.

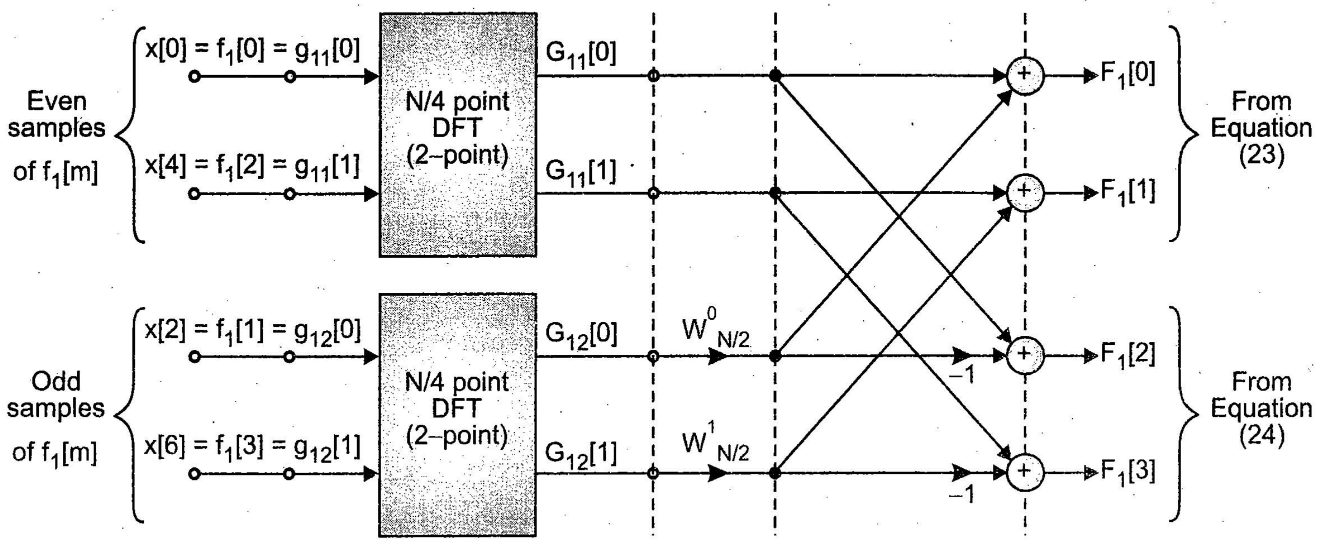


Fig. G-3 : $F_1(k)$, $\frac{N}{2}$ point DFT

Note that here,

$$\left.\begin{aligned} g_{11}(n) &= f_1(2m) = x(4n) = \{x(0), x(4)\} \\ \text{and} \quad g_{12}(n) &= f_1(2m+1) = x(4n+2) = \{x(2), x(6)\} \end{aligned}\right\} \quad \ldots(25)$$

Now similar to Equations (21) and (22) we can write equations for $F_2(k)$ as follows :

$$F_2(k) = G_{21}(k) + W_{N/2}^{k} G_{22}(k), \qquad k = 0, 1, \ldots \frac{N}{4} - 1 \quad \ldots(26)$$

$$\text{and} \quad F_2\left(k + \frac{N}{4}\right) = G_{21}(k) - W_{N/2}^{k} G_{22}(k), \qquad k = 0, 1, \ldots, \frac{N}{4} - 1 \quad \ldots(27)$$

Here $G_{21}(k)$ is DFT of $g_{21}(n)$ and $G_{22}(k)$ is DFT of $g_{22}(n)$. The values of K are 0 and 1. Putting these values in Equation (26),

$$\left.\begin{aligned} F_2(0) &= G_{21}(0) + W_{N/2}^{0} G_{22}(0) \\ F_2(1) &= G_{21}(1) + W_{N/2}^{1} G_{22}(1) \end{aligned}\right\} \quad \ldots(28)$$

Similarly from Equation (27) we get,

$$\left.\begin{aligned} F_2\left(0+\frac{8}{4}\right) &= F_2(2) = G_{21}(0) - W_{N/2}^{0}\, G_{22}(0) \\ F_2\left(1+\frac{8}{4}\right) &= F_2(3) = G_{21}(1) - W_{N/2}^{1}\, G_{22}(1) \end{aligned}\right\} \qquad ...(29)$$

The graphical representation of Equations (28) and (29) is shown in Fig. G-4.

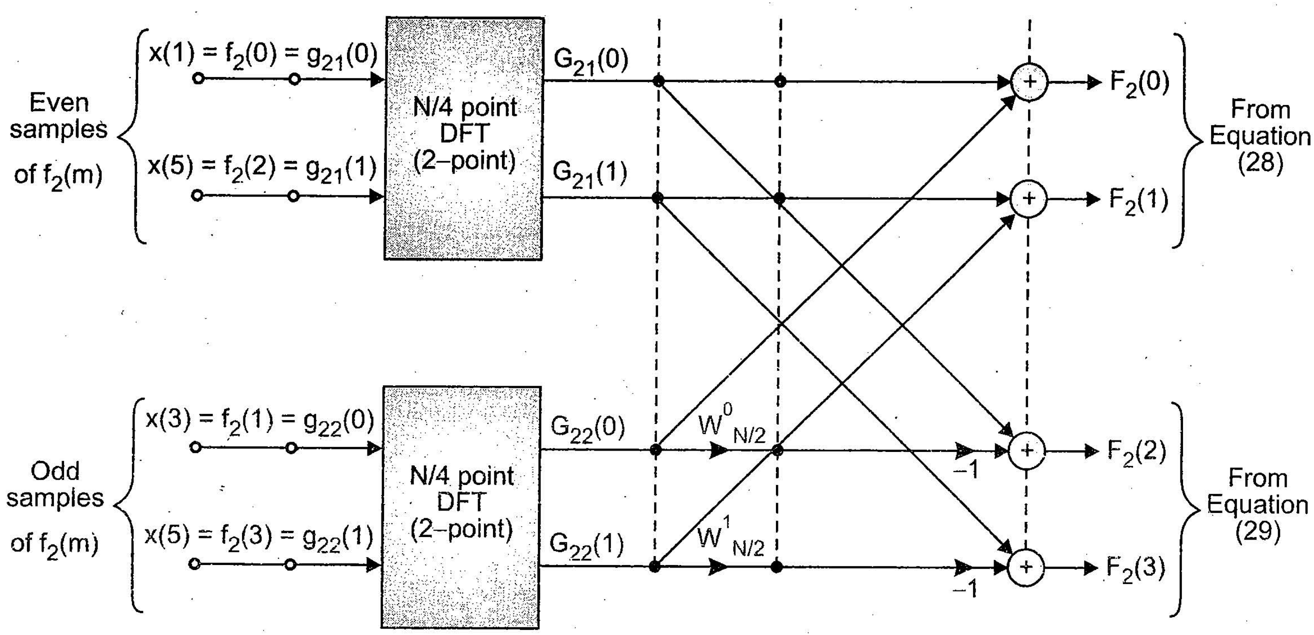


Fig. G-4 : $F_2(k)$, $\frac{N}{2}$ point DFT

Note that here,

$$\left.\begin{aligned} g_{21}(n) &= f_2(2n) = x(4n+1) = \{x(1), x(5)\} \\ g_{22}(n) &= f_2(2n+1) = x(4n+3) = \{x(3), x(7)\} \end{aligned}\right\} \qquad ...(30)$$

Combination of first and second stage of decimation :

Combining Fig. G-3 and Fig. G-4 in Fig. G-2 we get the combination of first and second stage of decimation. It is shown in Fig. G-5.

At this stage we have $\frac{N}{4}$ that means 2 point sequences. So further decimation is not possible. As shown in Fig. G-5; we have to compute 2-point DFT.

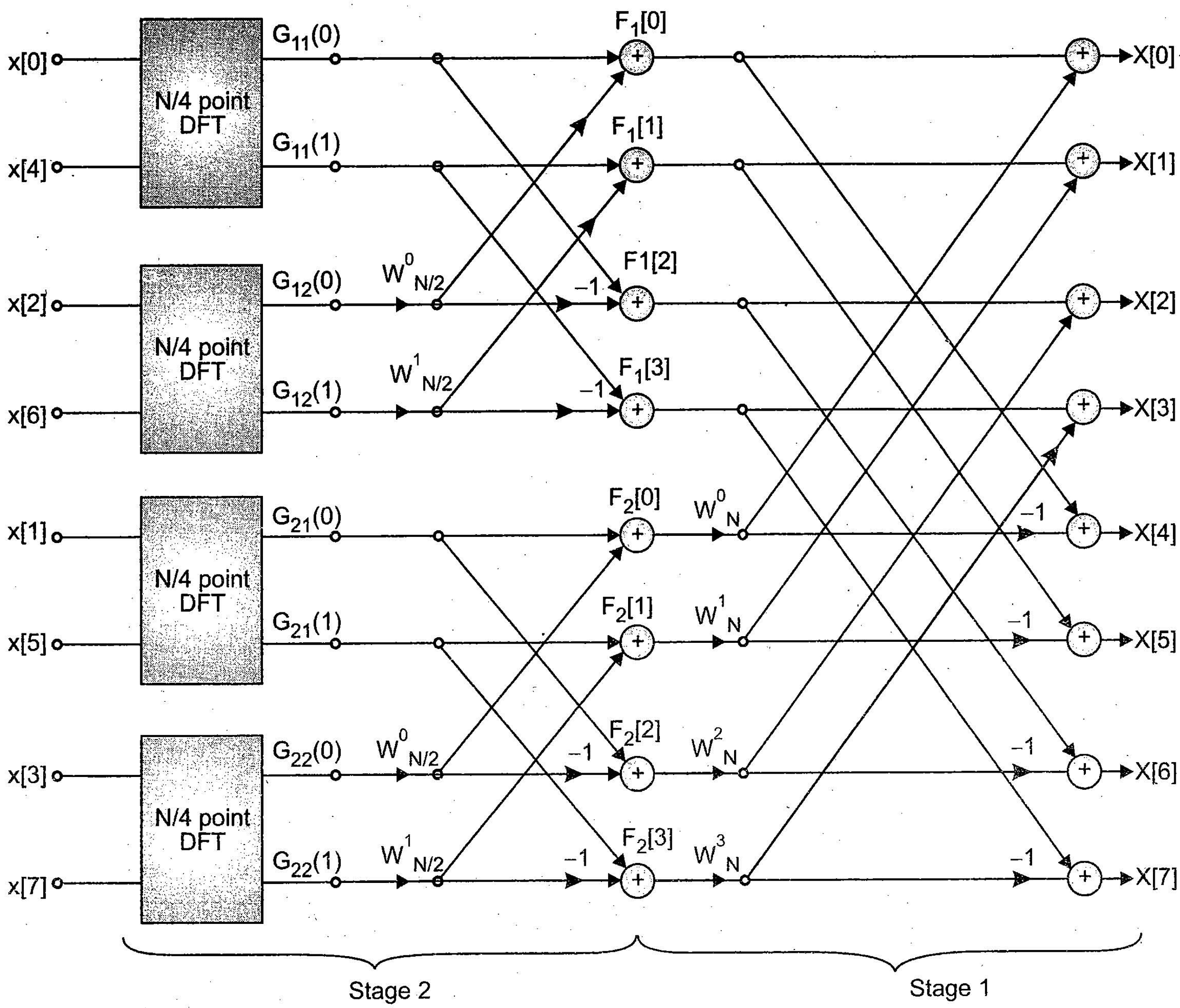


Fig. G-5 : Combination of first and second stage of decimation

Computation of 2-point DFT : According to the basic definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, ..., N-1 \qquad ...(31)$$

We will use Equation (31) to compute 2-point DFT. From Fig. G-6, consider the first block of 2-point DFT. It is separately drawn as shown in Fig. G-6.

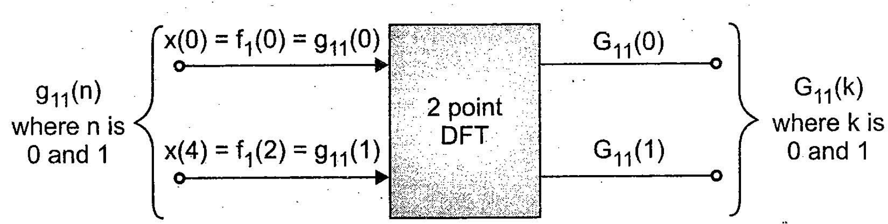


Fig. G-6 : Block of 2-point DFT

Here input sequences are $g_{11}(0)$ and $g_{11}(1)$. We can denote it by $g_{11}(n)$; where n varies from 0 to 1. Now the output sequences are $G_{11}(0)$ and $G_{11}(1)$. We can denote it by $G_{11}(k)$; where 'k' varies from 0 to 1. Here $G_{11}(k)$ is DFT of $g_{11}(n)$.

Thus for $G_{11}(k)$ we can write Equation (31) as,

$$G_{11}(k) = \sum_{n=0}^{1} g_{11}(n) W_2^{kn}, \quad k = 0, 1 \qquad ...(32)$$

Note that this is 2 point DFT, so we have put N = 2.

Now putting values of k in Equation (32) we get,

$$\text{For } k = 0 \Rightarrow G_{11}(0) = \sum_{n=0}^{1} g_{11}(n) W_2^0$$

$$\text{But } W_2^0 = 1$$

$$\therefore \quad G_{11}(0) = \sum_{n=0}^{1} g_{11}(n)$$

Expanding the summation we get,

$$\boxed{G_{11}(0) = g_{11}(0) + g_{11}(1)} \qquad ...(33)$$

$$\text{For } k = 1 \Rightarrow G_{11}(1) = \sum_{n=0}^{1} g_{11}(n) W_2^n$$

Expanding the summation we get,

$$G_{11}(1) = g_{11}(0) W_2^0 + g_{11}(1) W_2^1 \qquad ...(34)$$

$$\text{We have} \quad W_N = e^{-\frac{j2\pi}{N}}$$

$$\therefore \quad W_2^1 = \left(e^{-\frac{j2\pi}{2}}\right)^1 = e^{-j\pi} = \cos\pi - j\sin\pi = -1 - j0$$

$$\therefore \quad W_2^1 = -1 \qquad \text{And} \quad W_2^0 = 1$$

Putting these values in Equation (34) we get,

$$\boxed{G_{11}(1) = g_{11}(0) - g_{11}(1)} \qquad ...(35)$$

Using Equations (33) and (35), we can represent the computation of 2-point DFT as shown in Fig. G-7. This structure looks like a butterfly. So it is called as FFT butterfly structure.

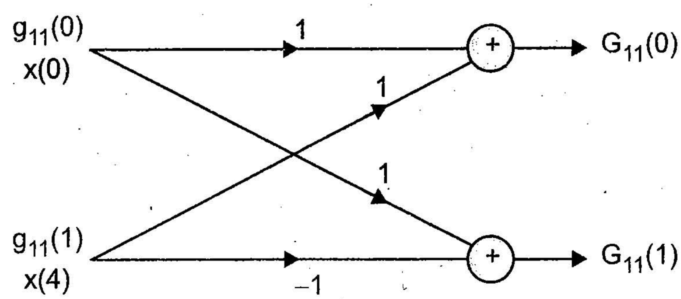


Fig. G-7 : FFT butterfly structure

Now we know that $W_2^0 = 1$. Thus we can modify Equation (33) and (35) as follows :

$$G_{11}(0) = g_{11}(0) + W_2^0 g_{11}(1) \quad \text{...(36)}$$

and $$G_{11}(1) = g_{11}(0) - W_2^0 g_{11}(1) \quad \text{...(37)}$$

This modified butterfly structure is shown in Fig. G-8.

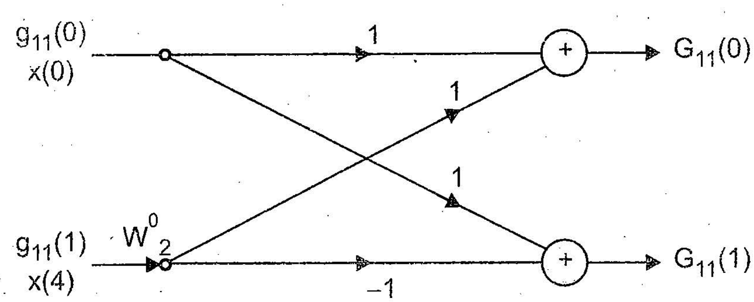


Fig. G-8 : Modified butterfly structure

Similarly for other 2-point DFTs we can draw the butterfly structure.

Total signal flow-graph for 8-point DIT FFT :

The total signal flow graph is obtained by interconnecting all stages of decimation. In this case, it is obtained by interconnecting first and second stage of decimation. But the starting block is the block used to compute 2-point DFT (butterfly structure). The total signal flow graph is shown in Fig. G-9.

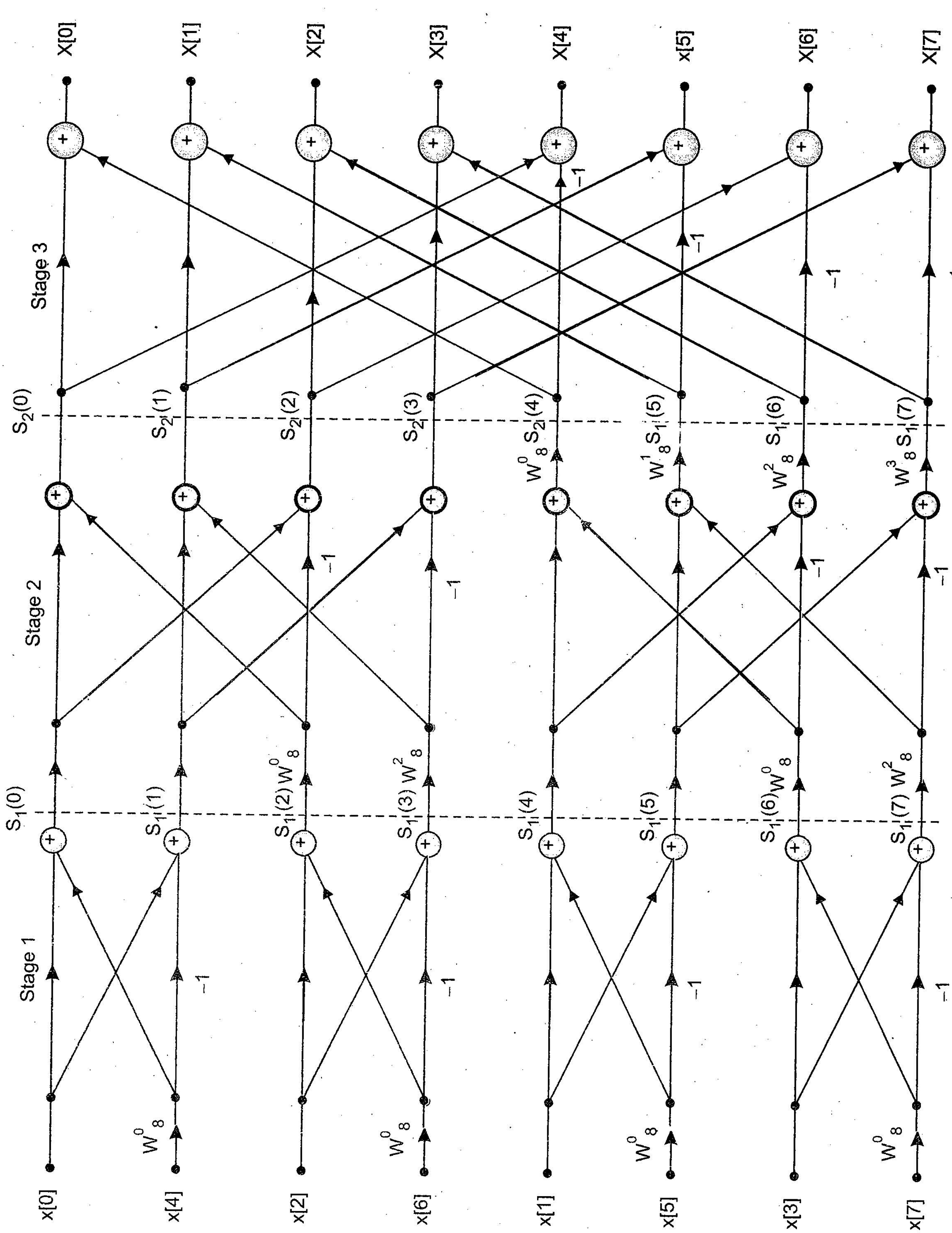


Fig. G-9 : Total signal flow graph for 8 point DIT FFT

Prob. 1 : Compute the eight-point DFT of a sequence.

$x(n) = \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\right\}$ Using in-place radix-2 decimation in time FFT algorithm.

Soln. : This flow graph is shown in Fig. G-10.

Here $s_1(n)$ represents output of stage - 1 and $s_2(n)$ represents output of stage - 2. The different values of twiddle factor are

$$W_8^0 = e^0 = 1$$

$$W_8^1 = e^{-j\frac{\pi}{4}} = 0.707 - j\,0.707$$

$$W_8^2 = e^{-j\frac{\pi}{2}} = -j$$

$$W_8^3 = -0.707 - j\,0.707$$

Output of stage - 1 :

$$s_1(0) = x(0) + W_8^0 x(4) = \frac{1}{2} + 1(0) = \frac{1}{2}$$

$$s_1(1) = x(0) - W_8^0 x(4) = \frac{1}{2} - 1(0) = \frac{1}{2}$$

$$s_1(2) = x(2) + W_8^0 x(6) = \frac{1}{2} + 1(0) = \frac{1}{2}$$

$$s_1(3) = x(2) - W_8^0 x(6) = \frac{1}{2} - 1(0) = \frac{1}{2}$$

$$s_1(4) = x(1) + W_8^0 x(5) = \frac{1}{2} + 1 \cdot (0) = \frac{1}{2}$$

$$s_1(5) = x(1) - W_8^0 x(5) = \frac{1}{2} - 1 \cdot (0) = \frac{1}{2}$$

$$s_1(6) = x(3) + W_8^0 x(7) = \frac{1}{2} + 1 \cdot (0) = \frac{1}{2}$$

$$s_1(7) = x(3) - W_8^0 x(7) = \frac{1}{2} - 1 \cdot (0) = \frac{1}{2}$$

Output of stage - 2 :

$$s_2(0) = s_1(0) + W_8^0 s_1(2) = \frac{1}{2} + 1 \cdot \left(\frac{1}{2}\right) = 1$$

$$s_2(1) = s_1(1) + W_8^2 s_1(3) = \frac{1}{2} - j\frac{1}{2}$$

$$s_2(2) = s_1(0) - W_8^0 s_1(2) = \frac{1}{2} - \frac{1}{2} = 0$$

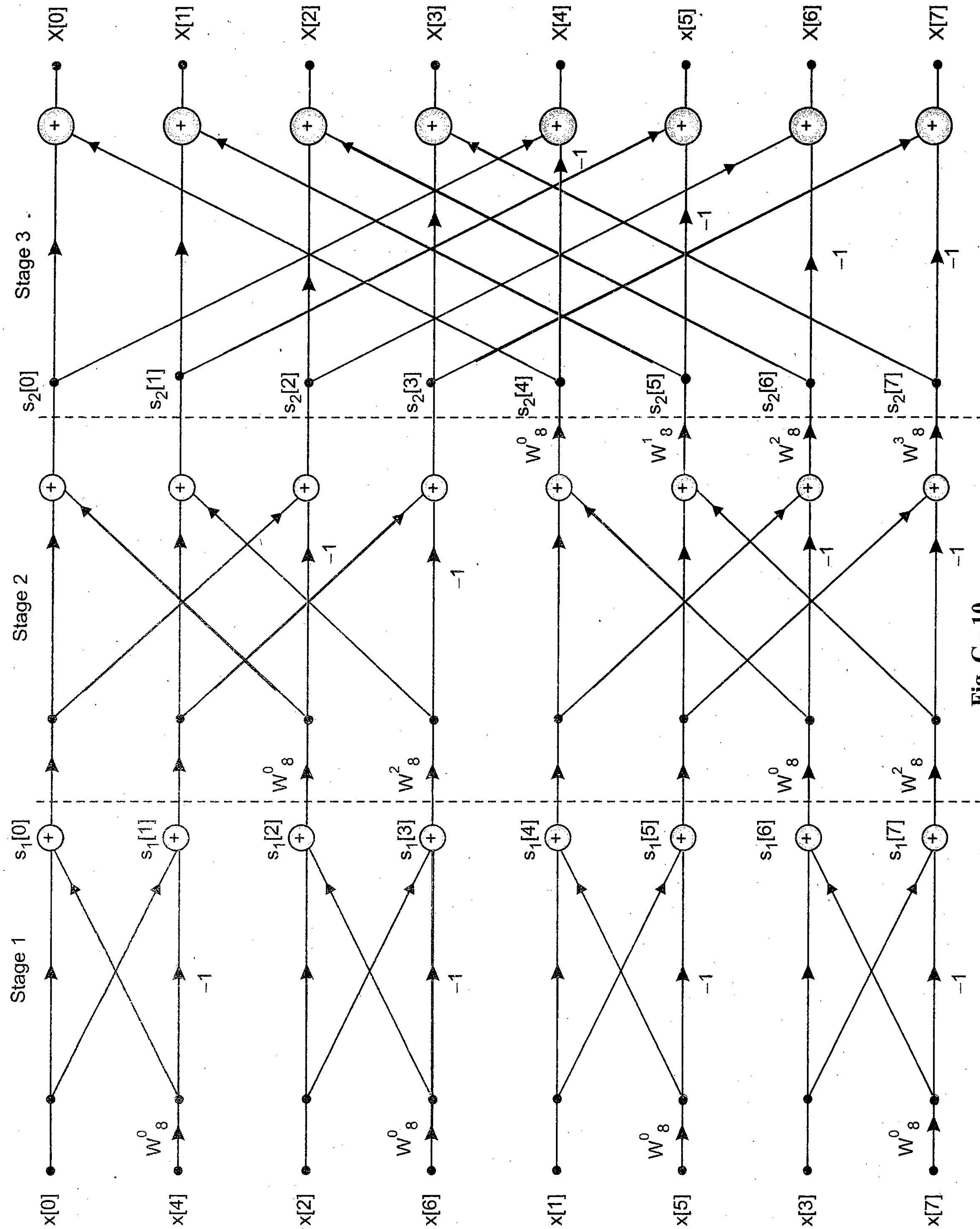

Stage 1
Stage 2
Stage 3
x[0]
x[4]
x[2]
x[6]
x[1]
x[5]
x[3]
x[7]
s1[0]
s1[1]
s1[2]
s1[3]
s1[4]
s1[5]
s1[6]
s1[7]
s2[0]
s2[1]
s2[2]
s2[3]
s2[4]
s2[5]
s2[6]
s2[7]
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]

Fig. G - 10

$$s_2(3) = s_1(1) - W_8^2 s_1(3) = \frac{1}{2} + j\frac{1}{2}$$

$$s_2(4) = s_1(4) + W_8^0 s_1(6) = \frac{1}{2} + \frac{1}{2} = 1$$

$$s_2(5) = s_1(5) + W_8^2 s_1(7) = \frac{1}{2} - j\frac{1}{2}$$

$$s_2(6) = s_1(4) - W_8^0 s_1(6) = \frac{1}{2} - \frac{1}{2} = 0$$

$$s_2(7) = s_1(5) - W_8^2 s_1(7) = \frac{1}{2} + j\frac{1}{2}$$

Final output :

$$X(0) = s_2(0) + W_8^0 s_2(4) = 1 + 1 = 2$$

$$X(1) = s_2(1) + W_8^1 s_2(5) = \left(\frac{1}{2} - j\frac{1}{2}\right) + (0.707 - j\,0.707)\left(\frac{1}{2} - j\frac{1}{2}\right)$$

$$= 0.5 - j\,1.207$$

$$X(2) = s_2(2) + W_8^2 s_2(6) = 0 + (-j)(0) = 0$$

$$X(3) = s_2(3) + W_8^3 s_2(7) = \left(\frac{1}{2} + j\frac{1}{2}\right) + (-0.707 - j\,0.707)\left(\frac{1}{2} + j\frac{1}{2}\right)$$

$$= \left(\frac{1}{2} + j\frac{1}{2}\right) + (0 - j\,0.707) = 0.5 - j\,0.207$$

$$X(4) = s_2(0) - W_8^0 s_2(4) = 1 - 1.1 = 0$$

$$X(5) = s_2(1) - W_8^1 s_2(5)$$

$$= \left(\frac{1}{2} - j\frac{1}{2}\right) - (0.707 - j\,0.707)\left(\frac{1}{2} - j\frac{1}{2}\right)$$

$$= \left(\frac{1}{2} - j\frac{1}{2}\right) - (-0.707\,j)$$

$$\therefore \quad X(5) = 0.5 + j\,0.207$$

$$X(6) = s_2(2) - W_8^2 s_2(6) = 0 + j \cdot (0) = 0$$

$$X(7) = s_2(3) - W_8^3 s_2(7)$$

$$= \left(\frac{1}{2} + j\frac{1}{2}\right) - (-0.707 - j\,0.707)\left(\frac{1}{2} + j\frac{1}{2}\right)$$

$$= \left(\frac{1}{2} + j\frac{1}{2}\right) + 0.707\,j = 0.5 + j\,1.21$$

Thus,

$$X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$$

$$\therefore \; X(k) = \{2,\ 0.5 - j\,1.207,\ 0,\ 0.5 - j\,0.207,\ 0,\ 0.5 + j\,0.207,\ 0,\ 0.5 + j\,1.21\}$$

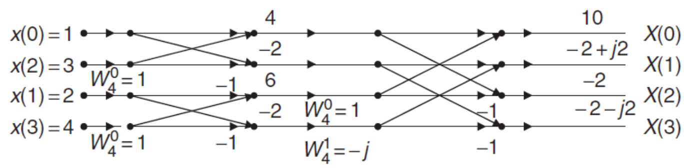
Example 2

Given a sequence $x(n)$ for $0 \le n \le 3$, where $x(0) = 1$, $x(1) = 2$, $x(2) = 3$, and $x(3) = 4$,

a. Evaluate its DFT $X(k)$ using the decimation-in-time FFT method.

Solution:

a. Using the block diagram in Figure 4.34 leads to



Prob. 3 : Derive DIT FFT flow graph for N = 4 hence find DFT of x (n) = {1, 2, 3, 4}

Soln. :

First stage of decimation :

We have the equations for first stage of decimation.

$$X(k) = F_1(k) + W_N^k F_2(k), \qquad k = 0, 1 \dots \frac{N}{2} - 1 \qquad \dots(1)$$

and $$X\left(k + \frac{N}{2}\right) = F_1(k) - W_N^k F_2(k), \qquad k = 0, 1, \dots \frac{N}{2} - 1 \qquad \dots(2)$$

Here N = 4

$$\therefore \quad X(k) = F_1(k) + W_4^k F_2(k), \qquad k = 0, 1 \qquad \dots(3)$$

and $$X(k+2) = F_1(k) - W_4^k F_2(k), \qquad k = 0, 1 \qquad \dots(4)$$

Putting values of k in Equation (3) we get,

$$\left.\begin{aligned} X(0) &= F_1(0) + W_4^0 F_2(0) \\ \text{and} \quad X(1) &= F_1(1) + W_4^1 F_2(1) \end{aligned}\right\} \qquad \dots(5)$$

Similarly putting values of k in Equation (4) we get,

$$\left.\begin{aligned} X(2) &= F_1(0) - W_4^0 F_2(0) \\ \text{and} \quad X(3) &= F_1(1) - W_4^1 F_2(1) \end{aligned}\right\} \qquad \dots(6)$$

This signal flow graph is shown in Fig. G-12(a)

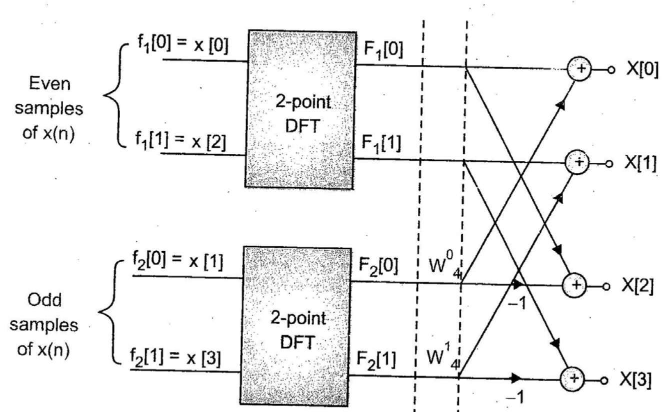


Fig. G-12(a)

Now we will replace each 2-point DFT by butterfly structure as shown in Fig. G-12(b).

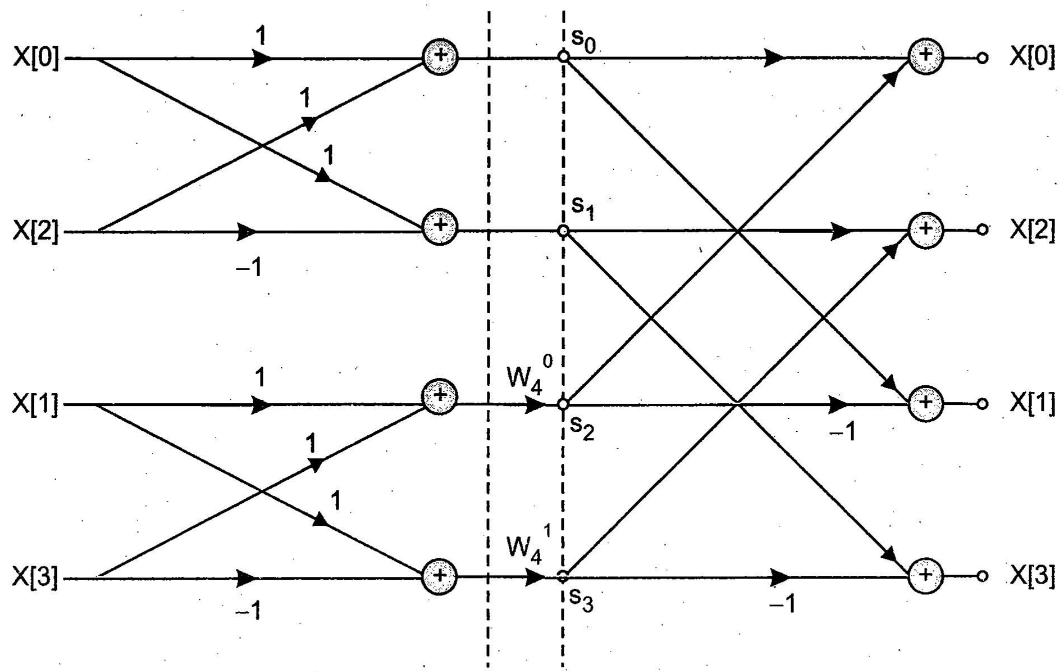


Fig. G-12(b)

The given sequence is,

$$x(n) = \{1, 2, 3, 4\}$$

The different values of twiddle factor are as follows :

$$W_4^0 = 1$$

$$W_4^1 = e^{-\frac{j2\pi}{4}\cdot 1} = e^{-\frac{j\pi}{2}}$$

$$= \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

The output s (n) is,

$$s_0 = x(0) + x(2) = 1 + 3 = 4$$

$$s_1 = x(0) - x(2) = 1 - 3 = -2$$

$$s_2 = [x(1) + x(3)]W_4^0 = 2 + 4 = 6$$

$$s_3 = [x(1) - x(3)]W_4^1 = (2-4)\cdot(-j) = 2j$$

The final output is,

$$X(0) = s_0 + s_2 = 4 + 6 = 10$$

$$X(1) = s_1 + s_3 = -2 + j2$$

$$X(2) = s_0 - s_2 = 4 - 6 = -2$$

$$X(3) = s_1 - s_3 = -2 - j2$$

Thus,

$$X(k) = \{X(0), X(1), X(2), X(3)\}$$

$$\therefore X(k) = \{10, -2 + j2, -2, -2 - j2\}$$

Prob. 4 : Draw flow diagram of DITFFT for N = 16.

Soln. :

(1) Here N = 16, means it is 16 point DFT.

(2) Total number of stages = 4

(3) The first stage of decimation using two 8-point DFT is shown in Fig. G-13(a).

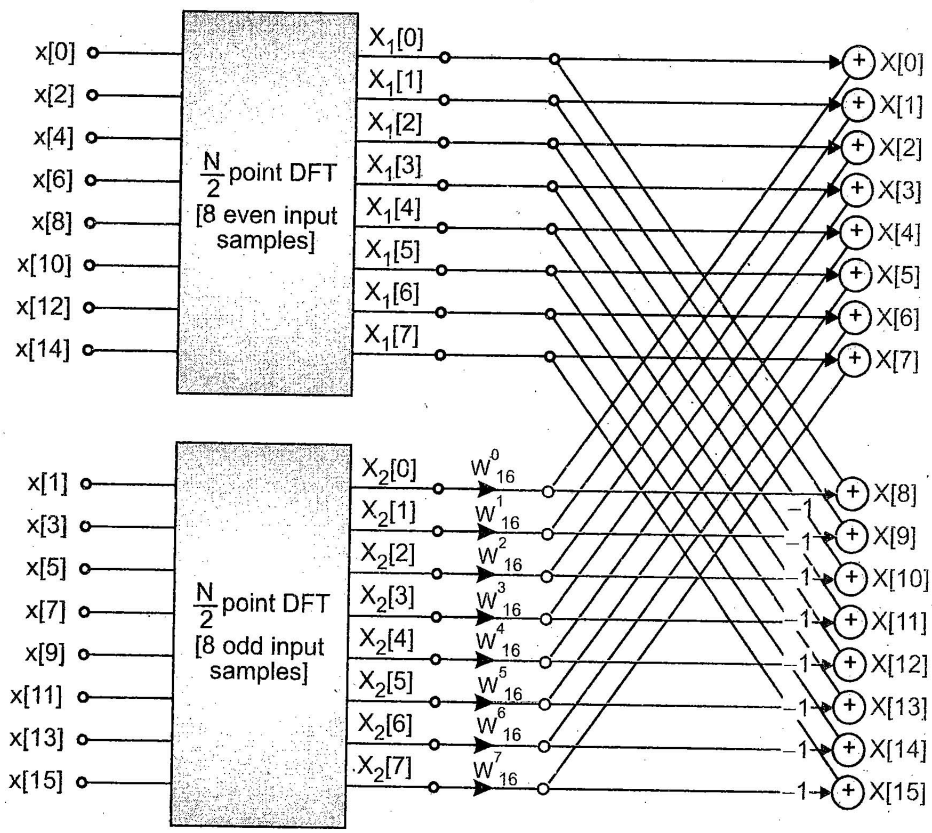


Fig. G-13(a)

(4) In the second stage each 8 point DFT is divided into 2 four point DFTs as shown in Fig. G-13(b).

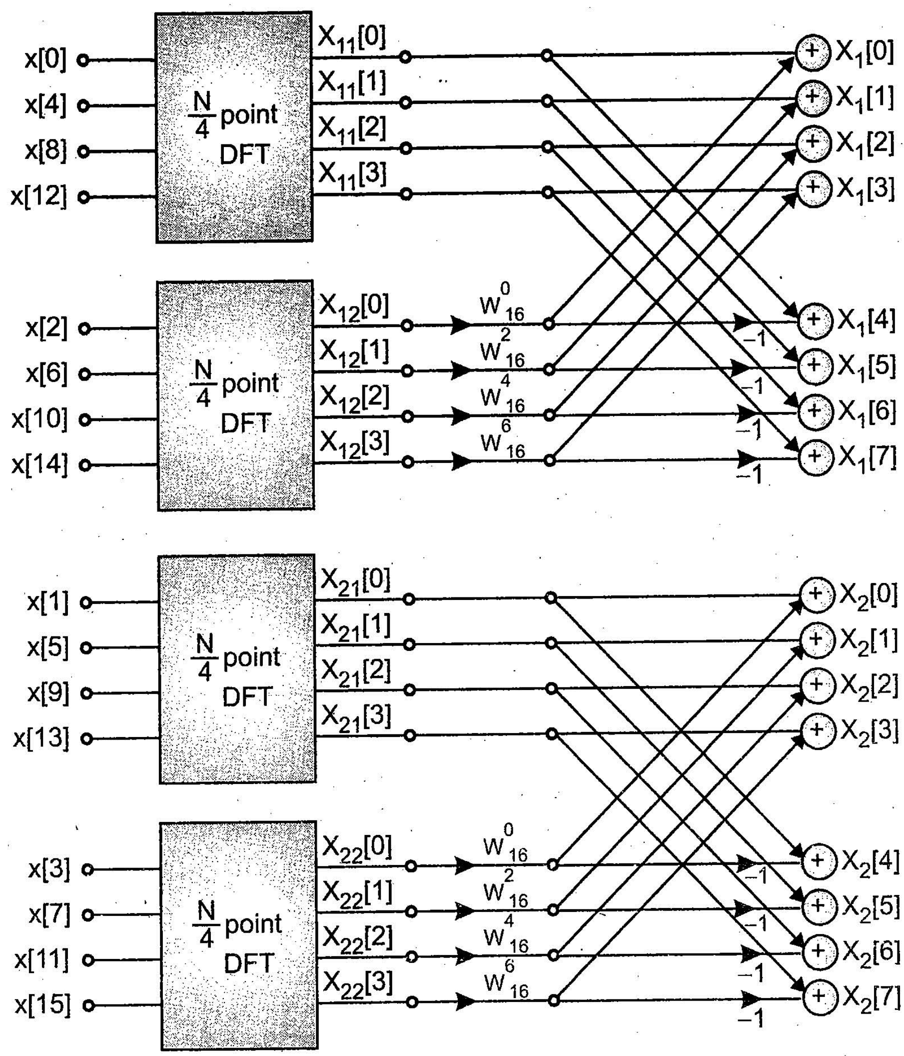
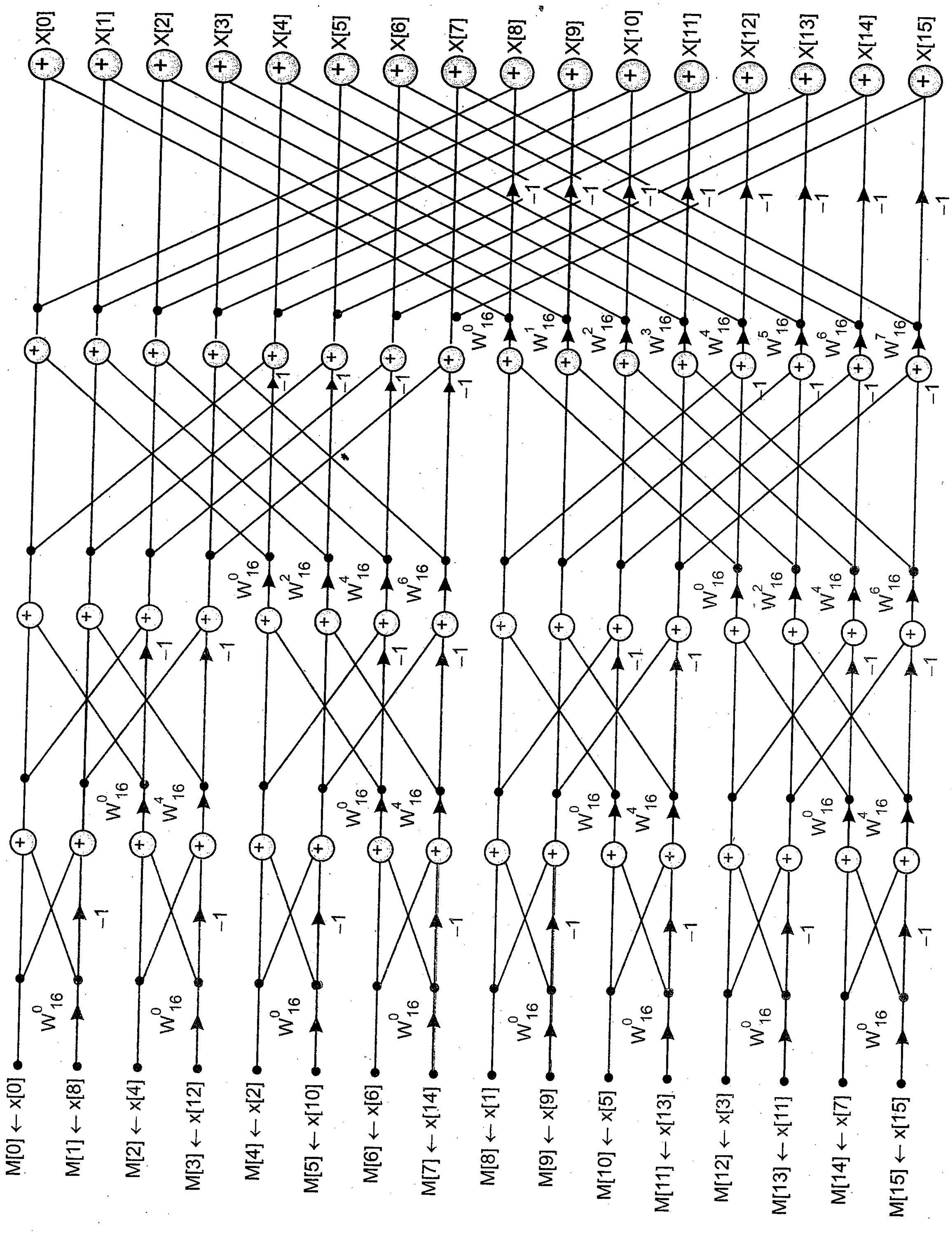


Fig. G-13(b)



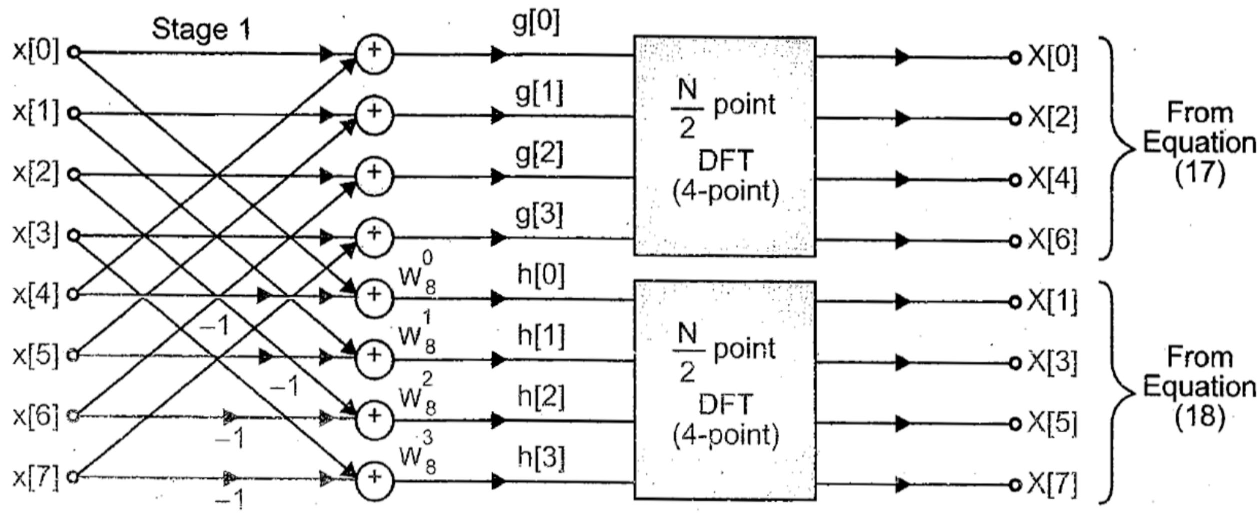
M[0] ← x[0]
M[1] ← x[8]
M[2] ← x[4]
M[3] ← x[12]
M[4] ← x[2]
M[5] ← x[10]
M[6] ← x[6]
M[7] ← x[14]
M[8] ← x[1]
M[9] ← x[9]
M[10] ← x[5]
M[11] ← x[13]
M[12] ← x[3]
M[13] ← x[11]
M[14] ← x[7]
M[15] ← x[15]
X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
X[7]
X[8]
X[9]
X[10]
X[11]
X[12]
X[13]
X[14]
X[15]

Fig. G-13(c)

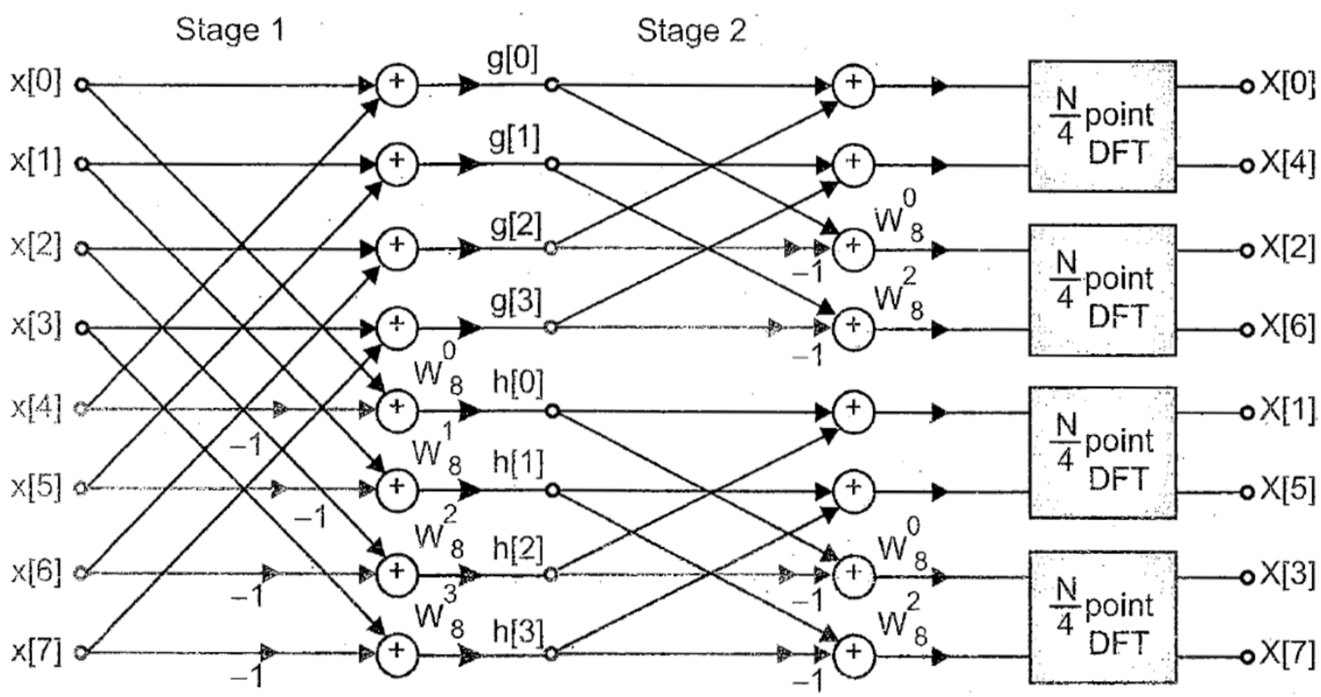
Radix-2 Decimation In Frequency (DIF) FFT Algorithm :

Decimation in frequency stands for splitting the sequences in terms of frequency. That means we have to split output sequences into smaller subsequences. This decimation is done as follows :

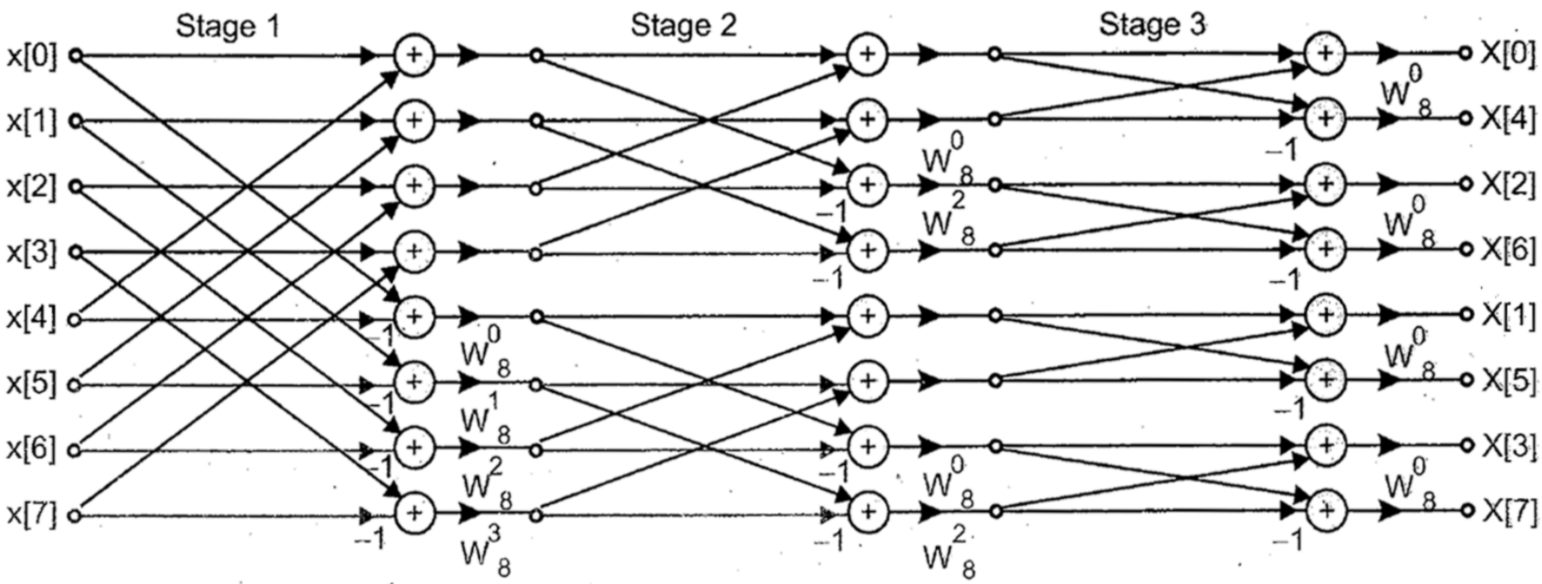
This algorithm is similar to the DITFFT but here we split k in to even and odd sequences (k=2r and k=2r+1). We will skip the derivation of this algorithm and jump to the final form and how to use it to calculate the DFT.



First stage of decimation



Second stage of decimation



Total flow graph for 8-point DIF-FFT

The computational complexity and the memory requirement is same as that of DIT-FFT.

Prob. 1 : Obtain DFT of a sequence

$$x(n) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\right)$$

Using decimation in frequency FFT algorithm.

Soln. : The total flow graph is shown in Fig. G-18.

Here g (n) is output of first stage.

h (n) is output of second stage.

The values of twiddle factor are as follows :

$$W_8^0 = e^0 = 1$$

$$W_8^1 = e^{-\frac{j\pi}{4}} = 0.707 - j\,0.707$$

$$W_8^2 = e^{\frac{-j\pi}{2}} = -j$$

$$W_8^3 = -0.707 - j\,0.707$$

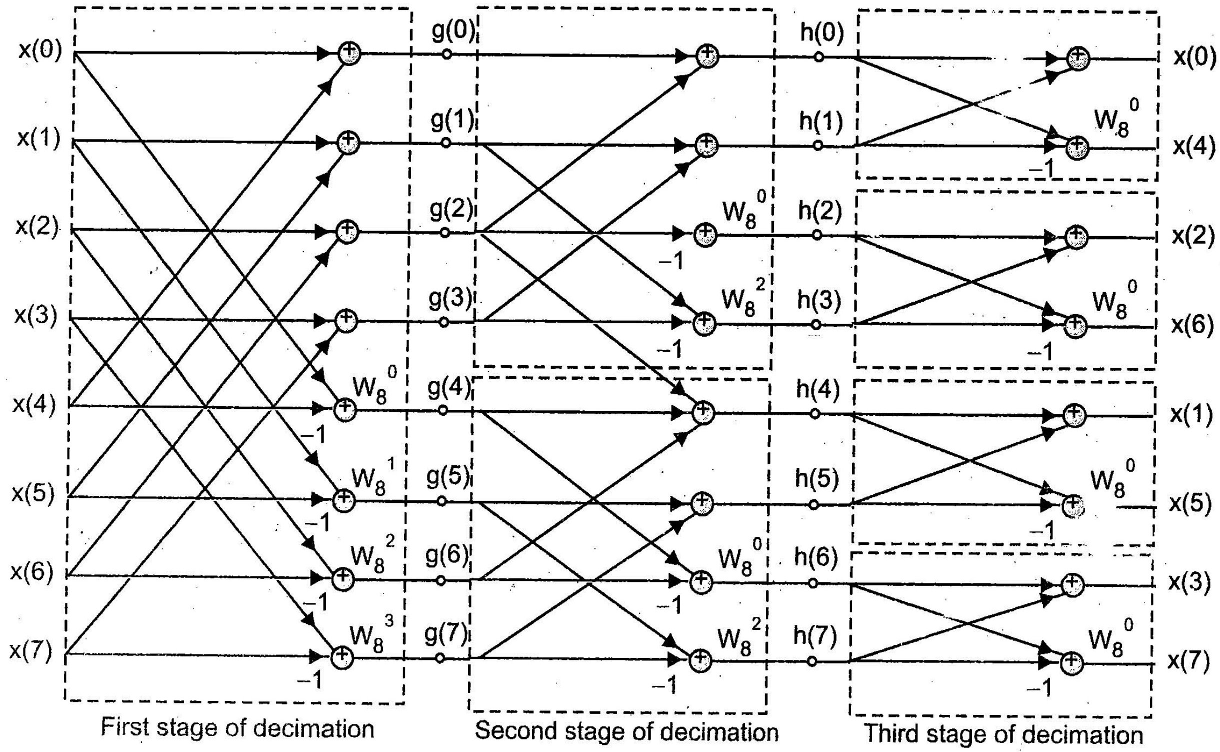


Fig. G-18

Output of stage - 1 :

$$g(0) = x(0)+x(4) = \frac{1}{2}+0 = 0.5$$

$$g(1) = x(1)+x(5) = \frac{1}{2}+0 = 0.5$$

$$g(2) = x(2)+x(6) = \frac{1}{2}+0 = 0.5$$

$$g(3) = x(3)+x(7) = \frac{1}{2}+0 = 0.5$$

$$g(4) = [x(0)-x(4)]\,W_8^0 = \left[\frac{1}{2}-0\right]1 = 0.5$$

$$g(5) = [x(1)-x(5)]\,W_8^1 = \left[\frac{1}{2}-0\right](0.707-j\,0.707)$$

$$= 0.3535-j\,0.3535$$

$$g(6) = [x(2)-x(6)]\,W_8^2 = \left[\frac{1}{2}-0\right](-j) = -j\,0.5$$

$$g(7) = [x(3)-x(7)]\,W_8^3 = \left[\frac{1}{2}-0\right](-0.707-j\,0.707)$$

$$= -0.3535-j\,0.3535$$

Output of stage - 2 :

$$h(0) = g(0)+g(2) = 0.5 + 0.5 = 1$$

$$h(1) = [g(1)+g(3)] = (0.5+0.5) = 1$$

$$h(2) = [g(0)-g(2)]\,W_8^0 = (0.5-0.5)(+1) = 0$$

$$h(3) = [g(1)-g(3)]\,W_8^2 = (0.5-0.5)(-j) = 0$$

$$h(4) = g(4)+g(6) = 0.5-j\,0.5$$

$$h(5) = g(5)+g(7) = 0.3535-j\,0.3535-0.3535-j\,0.3535$$

$$\therefore\ h(5) = -j\,0.707$$

$$h(6) = [g(4)-g(6)]\,W_8^0 = [0.5+j\,0.5]\,1 = 0.5 + j\,0.5$$

$$h(7) = [g(5)-g(7)]\,W_8^2 = [0.3535-j\,0.3535+0.3535+j\,0.3535](-j)$$

$$\therefore\ h(7) = -j\,0.707$$

Final output :

$$X(0) = h(0)+h(1) = 1 + 1 = 2$$

$$X(1) = h(4)+h(5) = 0.5-j\,0.5-j\,0.707 = 0.5-j\,1.207$$

$$X(2) = h(2)+h(3) = 0 + 0 = 0$$

$$X(3) = [h(6)+h(7)]\,W_8^0 = [0.5+j\,0.5-j\,0.707]\cdot 1 = 0.5-j\,0.207$$

$$X(4) = [h(0)-h(1)]\,W_8^0 = [1-1]\cdot 1 = 0$$

$$X(5) = [h(4)-h(5)]\,W_8^0 = [(0.5-j\,0.5)+j\,0.707]\cdot 1$$

$$= 0.5+j\,0.207$$

$$X(6) = [h(2)-h(3)]\,W_8^0 = 0$$

$$X(7) = [h(6)-h(7)]\,W_8^0 = [0.5+j\,0.5+j\,0.707]$$

$$= 0.5+j\,1.21$$

$$\therefore\ X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$$

$$\boxed{\therefore\ X(k) = \{2, 0.5-j\,1.207, 0, 0.5-j\,0.207, 0, 0.5+j\,0.207, 0, 0.5+j\,1.21\}}$$

Prob. 2 : Using DIFFFT Find

DFT $X_1(k)$ of following sequence $x_1(n) = \{1, 2, -1, 2, 4, 2, -1, 2\}$

Soln. : The total flow graph is shown in Fig. G-19(a).

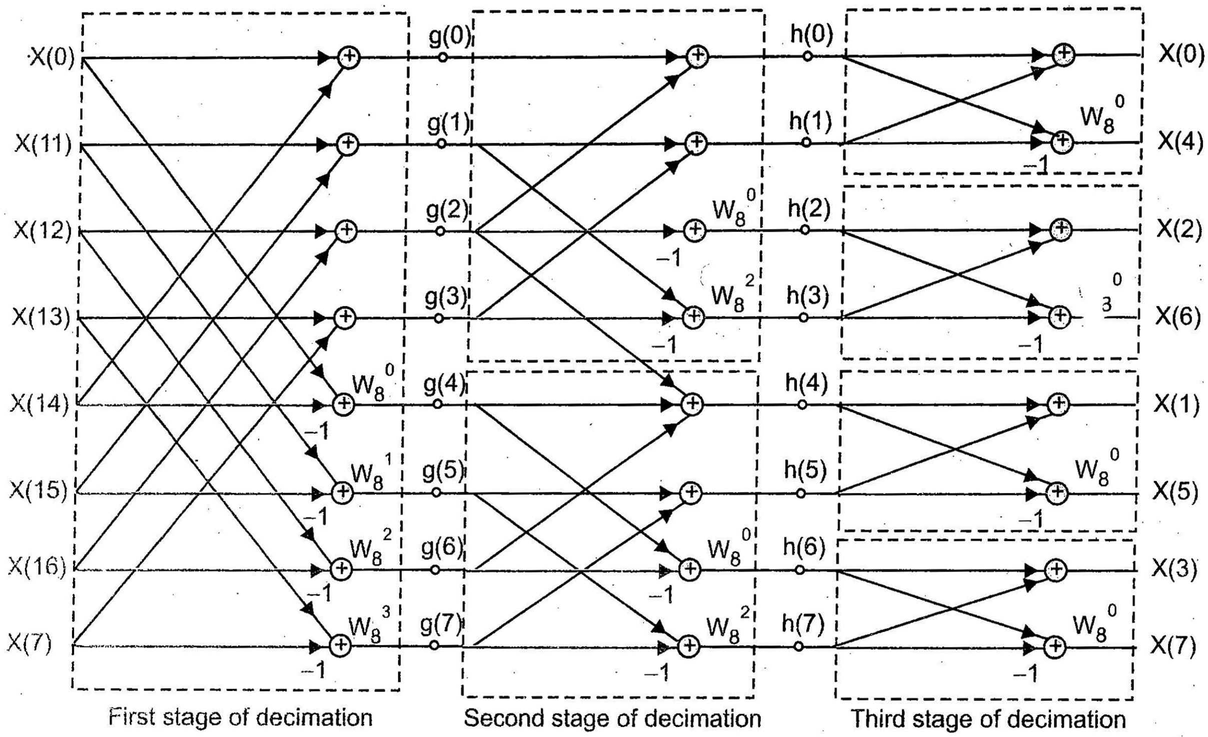


Fig. G-19(a)

The values of twiddle factor are as follows :

$$W_8^0 = 1$$

$$W_8^1 = 0.707 - j\,0.707$$

$$W_8^2 = -j$$

$$W_8^3 = -0.707 - j\,0.707$$

Here, $x(0) = 1$, $x(1) = 2$, $x(2) = -1$, $x(3) = 2$, $x(4) = 4$, $x(5) = 2$, $x(6) = -1$, $x(7) = 2$

Output of stage - 1 :

$$g(0) = x(0) + x(4) = 1 + 4 = 5$$

$$g(1) = x(1) + x(5) = 2 + 2 = 4$$

$$g(2) = x(2) + x(6) = -1 - 1 = -2$$

$$g(3) = x(3)+x(7) = 2+2 = 4$$
$$g(4) = [x(0)-x(4)]W_8^1 = [1-4]\times 1 = -3$$
$$g(5) = [x(1)-x(5)]W_8^1 = (2-2)W_8^1 = 0$$
$$g(6) = [x(2)-x(6)]W_8^2 = (-1+1)W_8^2 = 0$$
$$g(7) = [x(3)-x(7)]W_8^3 = (2-2)W_8^3 = 0$$

Output of stage - 2 :

$$h(0) = g(0)+g(2) = 5-2 = 3$$
$$h(1) = g(1)+g(3) = 4+4 = 8$$
$$h(2) = [g(0)-g(2)]W_8^0 = 5+2 = 7$$
$$h(3) = [g(1)-g(3)]W_8^2 = (4-4)(-j) = 0$$
$$h(4) = g(4)+g(6) = -3$$
$$h(5) = g(5)+g(7) = 0+0 = 0$$
$$h(6) = [g(4)-g(6)]W_8^0 = -3$$
$$h(7) = [g(5)-g(7)]W_8^2 = 0$$

Final output :

$$X(0) = h(0)+h(1) = 3+8 = 11$$
$$X(1) = h(4)+h(5) = -3+0 = -3$$
$$X(2) = h(2)+h(3) = 7+0 = 7$$
$$X(3) = [h(6)+h(7)]W_8^0 = -3$$
$$X(4) = [h(0)-h(1)]W_8^0 = 3-8 = -5$$
$$X(5) = [h(4)-h(5)]W_8^0 = -3$$
$$X(6) = [h(2)-h(3)]W_8^0 = 7$$
$$X(7) = [h(6)-h(7)]W_8^0 = -3$$

$$\therefore \quad X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$$

$$\boxed{\therefore \quad X(k) = \{11, -3, 7, -3, -5, -3, 7, -3\}}$$

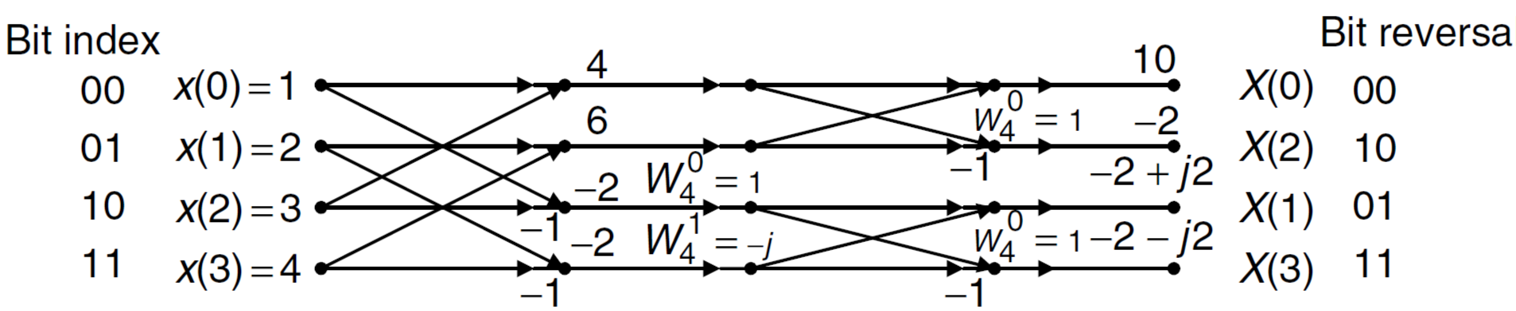
To remember the order of the input and output you can use the Bit-Reverse order, where you swap the bits horizontally while the middle bit remains the same.

Index mapping for fast Fourier transform.

Input Data	Index Bits	Reversal Bits	Output Data
$x(0)$	000	000	$X(0)$
$x(1)$	001	100	$X(4)$
$x(2)$	010	010	$X(2)$
$x(3)$	011	110	$X(6)$
$x(4)$	100	001	$X(1)$
$x(5)$	101	101	$X(5)$
$x(6)$	110	011	$X(3)$
$x(7)$	111	111	$X(7)$

Example: Given a sequence x(n) for 0 <= n <= 3, where x(0) = 1, x(1) = 2, x(2) = 3, and x(3) = 4, Evaluate its DFT X(k) using the **decimation-in-frequency** FFT method.

Sol:



Example: Given a sequence x(n) for 0 <= n <= 3, where x(0) = 1, x(1) = 2, x(2) = 3, and x(3) = 4, Evaluate its DFT X(k) using the **decimation-in-time** FFT method.

Sol: